

LOG CANONICAL MODELS FOR $\overline{\mathcal{M}}_{g,n}$

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ABSTRACT. We prove a formula of log canonical models for moduli space $\overline{\mathcal{M}}_{g,n}$ of pointed stable curves which describes all Hassett's moduli spaces of weighted pointed stable curves in a single equation. This is a generalization of the preceding result for genus zero to all genera.

1. INTRODUCTION

A central problem in algebraic geometry when studying a variety X is to determine all birational models of X . One way to approach this problem is to use divisors that have many sections. For example, for a big divisor D on X , one can hope to define and learn about a natural birational model

$$X(D) := \text{Proj} \bigoplus_{k \geq 0} H^0(X, \mathcal{O}(\lfloor kD \rfloor)).$$

Many results in birational geometry in last several decades are about overcoming of technical difficulties such as the finite generation of the section ring.

The moduli spaces $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ of stable curves and stable curves with marked points, are important as they give information about smooth curves and their degenerations. Moreover, as special varieties, they have played a useful role in illustrating and testing the goals of birational geometry for example the *minimal model program*.

In this paper we show the following theorem, as an example of log minimal model program applied to the moduli space $\overline{\mathcal{M}}_{g,n}$.

Theorem 1.1. *Let $\mathcal{A} = (a_1, a_2, \dots, a_n)$ be a weight datum satisfying $2g - 2 + \sum_{i=1}^n a_i > 0$. Then*

$$\overline{\mathcal{M}}_{g,n}(\mathcal{K}_{\overline{\mathcal{M}}_{g,n}} + 11\lambda + \sum_{i=1}^n a_i \psi_i) \cong \overline{\mathcal{M}}_{g,\mathcal{A}},$$

where $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is the coarse moduli space of Hassett's moduli space $\overline{\mathcal{M}}_{g,\mathcal{A}}$ of weighted pointed stable curves with weight datum \mathcal{A} ([Has03]).

This is proved for genus zero in [Moo11]. In this article we establish the result in all genera. A key step is to construct ample divisors on the moduli spaces $\overline{\mathcal{M}}_{g,\mathcal{A}}$, for $g > 0$ (Proposition 4.3).

To put these results into context, we next give some history of this problem. By [HM82, Har84], we know that for $g \geq 24$, the canonical divisor $\mathcal{K}_{\overline{\mathcal{M}}_g}$ is big, and by [BCHM10], that the canonical model $\overline{\mathcal{M}}_g(\mathcal{K}_{\overline{\mathcal{M}}_g})$ exists. The hope is that one will be able to describe

the canonical model as a moduli space itself, and this problem has attracted a great deal of attention.

One approach has been the *Hassett-Keel program*. By [CH88], it is well known that $\overline{\mathcal{M}}_g(K_{\overline{\mathcal{M}}_g} + D) \cong \overline{\mathcal{M}}_g$, where $D = \overline{\mathcal{M}}_g - \mathcal{M}_g$ is the sum of all boundary divisors. So if we figure out the log canonical models $\overline{\mathcal{M}}_g(K_{\overline{\mathcal{M}}_g} + \alpha D)$ for $0 \leq \alpha \leq 1$ and find the variation of log canonical models during we reduce the coefficient α from 1 to 0, we can finally obtain the canonical model. Still this problem is far from complete except small genera cases ([Has05, HL10]), we have understood many different compactifications of \mathcal{M}_g . For example, see [HH08, HH09, FS10].

We can perform a similar program for $\overline{\mathcal{M}}_{g,n}$, the moduli space of pointed stable curves. The first result in this direction is the thesis of M. Simpson ([Sim08]). He studied the log canonical model of $\overline{\mathcal{M}}_{0,n}$ assuming the F-conjecture. He proved that in a suitable range of β , $\overline{\mathcal{M}}_{0,n}(K_{\overline{\mathcal{M}}_{0,n}} + \beta D)$ is isomorphic to $\overline{\mathcal{M}}_{0,\mathcal{A}}$ where \mathcal{A} is symmetric weight datum which depends on β . This theorem was later proved without assuming the F-conjecture ([AS08, FS11, KM11]). For $g = 1$, there are results of Smyth ([Smy10, FS10]) considering birational models of type $\overline{\mathcal{M}}_{1,n}(s\lambda + t \sum_{i=1}^n \psi_i - D)$. In this case, birational models are given by moduli spaces of symmetric weighted curves with even worse singularities. In [Fed11] Fedorchuk showed that for any genus g and weight datum $\mathcal{A} = (a_1, a_2, \dots, a_n)$ satisfying $2g - 2 + \sum_{i=1}^n a_i > 0$, there is a divisor $D_{g,\mathcal{A}}$ on $\overline{\mathcal{M}}_{g,n}$ such that (1) $(\overline{\mathcal{M}}_{g,n}, D_{g,\mathcal{A}})$ is a *lc pair* and (2) $\overline{\mathcal{M}}_{g,n}(K_{\overline{\mathcal{M}}_{g,n}} + D_{g,\mathcal{A}}) \cong \overline{\mathcal{M}}_{g,\mathcal{A}}$.

Both formula in Theorem 1.1 and [Fed11] have their own interest. Theorem 1.1 says that the *same weight datum* determines the log canonical model of parameterized curves and that of the parameter space itself. Indeed, for any weight datum \mathcal{A} , there is a reduction morphism $\varphi_{\mathcal{A}} : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$ ([Has03, Theorem 4.1]). For a stable curve $(C, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n}$, its image $\varphi_{\mathcal{A}}(C)$ is given by the *log canonical model*

$$(1) \quad C(\omega_C + \sum_{i=1}^n a_i x_i) := \text{Proj} \bigoplus_{k \geq 0} H^0(C, \mathcal{O}([k(\omega_C + \sum_{i=1}^n a_i x_i)]))$$

of C .

In Section 2 we list the definition and computational results of several tautological divisors. In Section 3, we prove Proposition 3.1 which is the crucial step of the proof of Theorem 1.1. We give the rest of the proof in Section 4.

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2. A GLOSSARY OF DIVISORS ON $\overline{\mathcal{M}}_{g,\mathcal{A}}$

In this section, we recall definitions of tautological divisors on $\overline{\mathcal{M}}_{g,\mathcal{A}}$ and their push-forward/pull-back formulas. We omit proofs because they are results of simple generalizations of well-known results. For the proofs and ideas, see for instance [AC96, AC98, HM98, Has03, Fed11, FS11, Moo11].

Definition 2.1. Fix a weight datum $\mathcal{A} = (a_1, a_2, \dots, a_n)$. Let $[n] := \{1, 2, \dots, n\}$. For $I \subset [n]$, let $w_I := \sum_{i \in I} a_i$. Let $\pi : \mathcal{U}_{g,\mathcal{A}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$ be the universal family and $\sigma_i : \overline{\mathcal{M}}_{g,\mathcal{A}} \rightarrow$

$\mathcal{U}_{g,\mathcal{A}}$ for $i = 1, 2, \dots, n$ be the universal sections. Also let $\omega := \omega_{\mathcal{U}_{g,\mathcal{A}}/\overline{\mathcal{M}}_{g,\mathcal{A}}}$ be the relative dualizing sheaf.

- (1) The *kappa class*: $\kappa := \pi_*(c_1^2(\omega))$. Our definition is different from several others for example [AC96, AC98, ACG11].
- (2) The *Hodge class*: $\lambda := c_1(\pi_*(\omega))$.
- (3) The *psi classes*: For $i = 1, 2, \dots, n$, let \mathbb{L}_i be the line bundle on $\overline{\mathcal{M}}_{g,\mathcal{A}}$, whose fiber over $(C, x_1, x_2, \dots, x_n)$ is $\Omega_C|_{x_i}$, the cotangent space at x_i in C . The *i-th psi class* ψ_i is $c_1(\mathbb{L}_i)$. On the other hand, ψ_i can be defined in terms of intersection theory. $\psi_i = \pi_*(\omega \cdot \sigma_i) = -\pi_*(\sigma_i^2)$. The *total psi class* is $\psi := \sum_{i=1}^n \psi_i$.
- (4) *Boundaries of nodal curves*: Take a pair (j, I) for $0 \leq j \leq g$ and $I \subset [n]$. Suppose that if $j = 0$, then $w_I > 1$. Let $D_{j,I} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}$ be the closure of the locus of curves with two irreducible components $C_{j,I}$ and C_{g-j,I^c} such that $C_{j,I}$ (resp. C_{g-j,I^c}) is a smooth genus j (resp. $g-j$) curve and $x_i \in C_{j,I}$ if and only if $i \in I$. For a notational convenience, set $D_{j,I} = 0$ when $j = 0$ and $|I| \leq 1$. Let D_{irr} be the closure of the locus of irreducible nodal curves. Let D_{nod} be the sum of all $D_{j,I}$ and D_{irr} .
- (5) *Boundaries of curves with coincident sections*: Suppose that $I = \{i, j\}$ and $w_I \leq 1$. Let $D_{i=j}$ be the locus of curves with $x_i = x_j$. $D_{i=j}$ is equal to $\pi_*(\sigma_i \cdot \sigma_j)$. Let D_{sec} be the sum of all boundaries of curves with coincident sections.

The canonical divisor $K_{\overline{\mathcal{M}}_{g,\mathcal{A}}}$ is computed by Hassett ([Has03, Section 3.3.1]). By Mumford's relation $\kappa = 12\lambda - D_{\text{nod}}$ ([Mum77, Theorem 5.10]), it has two different presentations.

Lemma 2.2. [Has03, Section 3.3.1]

$$K_{\overline{\mathcal{M}}_{g,\mathcal{A}}} = \frac{13}{12}\kappa - \frac{11}{12}D_{\text{nod}} + \psi = 13\lambda - 2D_{\text{nod}} + \psi.$$

Next, we present the push-forward and pull-back formulas we will often use.

Let $\mathcal{A} = (a_1, a_2, \dots, a_n)$ and $\mathcal{B} = (b_1, b_2, \dots, b_n)$ be weight data such that $a_i \geq b_i$ for all $i = 1, 2, \dots, n$. For $I \subset [n]$, set $w_I^{\mathcal{A}} = \sum_{i \in I} a_i$ and $w_I^{\mathcal{B}} = \sum_{i \in I} b_i$.

Lemma 2.3. Let $\varphi_{\mathcal{A},\mathcal{B}} : \overline{\mathcal{M}}_{g,\mathcal{A}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{B}}$ be the reduction morphism ([Has03, Theorem 4.1]).

- (1) $\varphi_{\mathcal{A},\mathcal{B}*}(\kappa) = \kappa - \sum_{w_{\{j,k\}}^{\mathcal{A}} > 1, w_{\{j,k\}}^{\mathcal{B}} \leq 1} D_{j=k}.$
- (2) $\varphi_{\mathcal{A},\mathcal{B}*}(\lambda) = \lambda.$
- (3) $\varphi_{\mathcal{A},\mathcal{B}*}(\psi_i) = \psi_i + \sum_{w_{\{i,j\}}^{\mathcal{A}} > 1, w_{\{i,j\}}^{\mathcal{B}} \leq 1} D_{i=j}.$
- (4) $\varphi_{\mathcal{A},\mathcal{B}*}(D_{i,I}) = \begin{cases} 0, & i = 0, |I| \geq 3, w_I^{\mathcal{B}} \leq 1, \\ D_{j=k}, & i = 0, I = \{j, k\}, w_I^{\mathcal{B}} \leq 1, \\ D_{i,I}, & \text{otherwise.} \end{cases}$
- (5) $\varphi_{\mathcal{A},\mathcal{B}*}(D_{\text{irr}}) = D_{\text{irr}}.$
- (6) $\varphi_{\mathcal{A},\mathcal{B}*}(D_{j=k}) = D_{j=k}.$
- (7) $\varphi_{\mathcal{A},\mathcal{B}}^*(\kappa) = \kappa + \sum_{w_I^{\mathcal{B}} \leq 1, w_I^{\mathcal{A}} > 1} D_{0,I}.$
- (8) $\varphi_{\mathcal{A},\mathcal{B}}^*(\lambda) = \lambda.$

$$\begin{aligned}
(9) \quad \varphi_{\mathcal{A},\mathcal{B}}^*(\psi_i) &= \psi_i - \sum_{i \in I, w_I^{\mathcal{B}} \leq 1, w_I^{\mathcal{A}} > 1} D_{0,I}. \\
(10) \quad \varphi_{\mathcal{A},\mathcal{B}}^*(D_{i,I}) &= D_{i,I}. \\
(11) \quad \varphi_{\mathcal{A},\mathcal{B}}^*(D_{\text{irr}}) &= D_{\text{irr}}. \\
(12) \quad \varphi_{\mathcal{A},\mathcal{B}}^*(D_{j=k}) &= \begin{cases} D_{j=k} + \sum_{I \supset \{j,k\}, w_I^{\mathcal{B}} \leq 1} D_{0,I}, & w_{\{j,k\}}^{\mathcal{A}} \leq 1, \\ \sum_{I \supset \{j,k\}, w_I^{\mathcal{B}} \leq 1} D_{0,I}, & w_{\{j,k\}}^{\mathcal{A}} > 1. \end{cases}
\end{aligned}$$

The special case $\varphi_{(1,1,\dots,1),\mathcal{A}} : \overline{\mathcal{M}}_{g,(1,1,\dots,1)} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$ is particularly important. For notational convenience, let $\varphi_{\mathcal{A}} := \varphi_{(1,1,\dots,1),\mathcal{A}}$.

Corollary 2.4. *For $\varphi_{\mathcal{A}} : \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,(1,1,\dots,1)} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$,*

$$\begin{aligned}
(1) \quad \varphi_{\mathcal{A}*}(\kappa) &= \kappa - D_{\text{sec}}. \\
(2) \quad \varphi_{\mathcal{A}*}(\lambda) &= \lambda. \\
(3) \quad \varphi_{\mathcal{A}*}(\psi_i) &= \psi_i + \sum_{w_{\{i,j\}} \leq 1} D_{i=j}. \\
(4) \quad \varphi_{\mathcal{A}*}(D_{i,I}) &= \begin{cases} 0, & i = 0, |I| \geq 3, w_I \leq 1, \\ D_I, & i = 0, |I| = 2, w_I \leq 1, \\ D_{i,I}, & \text{otherwise.} \end{cases} \\
(5) \quad \varphi_{\mathcal{A}*}(D_{\text{irr}}) &= D_{\text{irr}}. \\
(6) \quad \varphi_{\mathcal{A}}^*(\kappa) &= \kappa + \sum_{w_I \leq 1} D_{0,I}. \\
(7) \quad \varphi_{\mathcal{A}}^*(\lambda) &= \lambda. \\
(8) \quad \varphi_{\mathcal{A}}^*(\psi_i) &= \psi_i - \sum_{i \in I, w_I \leq 1} D_{0,I}. \\
(9) \quad \varphi_{\mathcal{A}}^*(D_{i,I}) &= D_{i,I}. \\
(10) \quad \varphi_{\mathcal{A}}^*(D_{\text{irr}}) &= D_{\text{irr}}. \\
(11) \quad \varphi_{\mathcal{A}}^*(D_{j=k}) &= \sum_{I \supset \{j,k\}, w_I \leq 1} D_{0,I}.
\end{aligned}$$

Lemma 2.5. *Let $\rho : \overline{\mathcal{M}}_{g,\mathcal{A} \cup \{a_p\}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$ be the forgetful morphism ([Has03, Theorem 4.3]).*

$$\begin{aligned}
(1) \quad \rho^*(\kappa) &= \kappa + \sum_{w_{\{i,p\}} > 1} D_{0,\{i,p\}}. \\
(2) \quad \rho^*(\lambda) &= \lambda. \\
(3) \quad \rho^*(\psi_i) &= \begin{cases} \psi_i, & w_{\{i,p\}} \leq 1, \\ \psi_i - D_{0,\{i,p\}}, & w_{\{i,p\}} > 1. \end{cases} \\
(4) \quad \rho^*(D_{i,I}) &= D_{i,I} + D_{i,I \cup \{p\}}. \\
(5) \quad \rho^*(D_{\text{irr}}) &= D_{\text{irr}}. \\
(6) \quad \rho^*(D_{j=k}) &= \begin{cases} D_{j=k}, & w_{\{j,k,p\}} \leq 1, \\ D_{j=k} + D_{0,\{j,k,p\}}, & w_{\{j,k,p\}} > 1. \end{cases}
\end{aligned}$$

For $I = \{j_1, j_2, \dots, j_r\} \subset [n]$, let $D_{i,I}$ be a boundary of nodal curves. Set $I^c = \{k_1, k_2, \dots, k_s\}$. Then $D_{i,I}$ is isomorphic to $\overline{\mathcal{M}}_{i,\mathcal{A}_I} \times \overline{\mathcal{M}}_{g-i,\mathcal{A}_{I^c}}$ where $\mathcal{A}_I = (a_{j_1}, a_{j_2}, \dots, a_{j_r}, 1)$ and $\mathcal{A}_{I^c} =$

$(a_{k_1}, a_{k_2}, \dots, a_{k_s}, 1)$. Let $\eta_{i,I} : \overline{\mathcal{M}}_{i,A_I} \times \overline{\mathcal{M}}_{g-i,A_{I^c}} \cong D_{i,I} \hookrightarrow \overline{\mathcal{M}}_{g,A}$ be the inclusion. Let π_ℓ for $\ell = 1, 2$ be the projection from $\overline{\mathcal{M}}_{i,A_I} \times \overline{\mathcal{M}}_{g-i,A_{I^c}}$ to the ℓ -th component.

Lemma 2.6. *Let p (resp. q) be the last index of A_I (resp. A_{I^c}) with weight one.*

$$\begin{aligned}
(1) \quad & \eta_{i,I}^*(\kappa) = \pi_1^*(\kappa + \psi_p) + \pi_2^*(\kappa + \psi_q). \\
(2) \quad & \eta_{i,I}^*(\lambda) = \pi_1^*(\lambda) + \pi_2^*(\lambda). \\
(3) \quad & \eta_{i,I}^*(\psi_j) = \begin{cases} \pi_1^*(\psi_j), & j \in I, \\ \pi_2^*(\psi_j), & j \in I^c. \end{cases} \\
(4) \quad & \eta_{i,I}^*(D_{j,J}) = \begin{cases} -\pi_1^*(\psi_p) - \pi_2^*(\psi_q), & D_{i,I} = D_{j,J}, \\ \pi_1^*(D_{j,J}), & j \leq i, J \subset I, D_{i,I} \neq D_{j,J}, \\ \pi_1^*(D_{g-j,J^c}), & g-j \leq i, J^c \subset I, D_{i,I} \neq D_{j,J}, \\ \pi_2^*(D_{j,J}), & j \leq g-i, J \subset I^c, D_{i,I} \neq D_{j,J}, \\ \pi_2^*(D_{g-j,J^c}), & i \leq j, I \subset J, D_{i,I} \neq D_{j,J}, \\ 0, & \text{otherwise.} \end{cases} \\
(5) \quad & \eta_{i,I}^*(D_{\text{irr}}) = \pi_1^*(D_{\text{irr}}) + \pi_2^*(D_{\text{irr}}). \\
(6) \quad & \eta_{i,I}^*(D_{j=k}) = \begin{cases} \pi_1^*(D_{j=k}), & j, k \in I, \\ \pi_2^*(D_{j=k}), & j, k \notin I, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Let $(C, x_1, x_2, \dots, x_n, p, q)$ be a genus $g-1$, $\mathcal{A} \cup \{1, 1\}$ -stable curve. By gluing p and q , we obtain an \mathcal{A} -stable curve of genus g . Since this gluing operation is extended to families of curves and functorial, we obtain a morphism $\xi : \overline{\mathcal{M}}_{g-1, \mathcal{A} \cup \{1, 1\}} \rightarrow \overline{\mathcal{M}}_{g, \mathcal{A}}$. Moreover, ξ is an embedding and the image is precisely D_{irr} .

Lemma 2.7. *Let ξ be the gluing map and let p, q be two identified sections.*

$$\begin{aligned}
(1) \quad & \xi^*(\kappa) = \kappa + \psi_p + \psi_q. \\
(2) \quad & \xi^*(\lambda) = \lambda. \\
(3) \quad & \xi^*(\psi_i) = \psi_i. \\
(4) \quad & \xi^*(D_{i,I}) = D_{i,I} + D_{i-1, I \cup \{p, q\}}. \\
(5) \quad & \xi^*(D_{\text{irr}}) = D_{\text{irr}} - \psi_p - \psi_q + \sum_{p \in I, q \notin I} D_{i,I}. \\
(6) \quad & \xi^*(D_{j=k}) = D_{j=k}.
\end{aligned}$$

Finally, for a nonempty subset $I \subset [n]$, assume that $w_I \leq 1$. Let \mathcal{A}' be a new weight datum defined by replacing $(a_i)_{i \in I}$ with a single rational number $w_I = \sum_{i \in I} a_i$. We can define an embedding $\chi_I : \overline{\mathcal{M}}_{g, \mathcal{A}'} \rightarrow \overline{\mathcal{M}}_{g, \mathcal{A}}$ which sends an \mathcal{A}' -stable curve to the \mathcal{A} -stable curve obtained by replacing the point of weight w_I with $|I|$ points of weight $(a_i)_{i \in I}$ on the same position.

Lemma 2.8. *Let $\chi_I : \overline{\mathcal{M}}_{g, \mathcal{A}'} \rightarrow \overline{\mathcal{M}}_{g, \mathcal{A}}$ be the replacing morphism. Let p be the unique index of \mathcal{A}' replacing indices in I .*

$$\begin{aligned}
(1) \quad & \chi_I^*(\kappa) = \kappa. \\
(2) \quad & \chi_I^*(\lambda) = \lambda.
\end{aligned}$$

$$\begin{aligned}
(3) \quad \chi_I^*(\psi_i) &= \begin{cases} \psi_i, & i \notin I, \\ \psi_p, & i \in I. \end{cases} \\
(4) \quad \chi_I^*(D_{\text{nod}}) &= D_{\text{nod}}. \\
(5) \quad \chi_I^*(D_{\text{irr}}) &= D_{\text{irr}}. \\
(6) \quad \chi_I^*(D_{j=k}) &= \begin{cases} D_{j=k}, & j, k \notin I, \\ D_{j=p}, & j \notin I, k \in I, \\ -\psi_p, & j, k \in I. \end{cases}
\end{aligned}$$

3. A POSITIVITY RESULT ON FAMILIES OF CURVES

A key step of the proof of Theorem 1.1 is to construct an ample divisor on $\overline{\mathcal{M}}_{g,\mathcal{A}}$. In this section, we prove the following technical positivity result of a divisor, which will be used to the proof of the main theorem.

Proposition 3.1. *Fix a weight datum $\mathcal{A} = (a_1, a_2, \dots, a_n)$ and a positive genus g . Let B be an integral curve. Let $\pi : \mathcal{U} \rightarrow B$ be a flat family of \mathcal{A} -stable genus g curves and let $\sigma_i : B \rightarrow \mathcal{U}$ for $i = 1, 2, \dots, n$ be n sections. Suppose that a general fiber of π is smooth. Then there exists a positive rational number $\epsilon_{g,\mathcal{A}} > 0$ which depends only on g and \mathcal{A} such that*

$$(2) \quad (2\kappa + \psi) \cdot B \geq \epsilon_{g,\mathcal{A}} \cdot \text{mult}_x B$$

for any point $x \in B$.

Remark 3.2. (1) Proposition 3.1 does *not* imply that $2\kappa + \psi$ is ample on $\overline{\mathcal{M}}_{g,\mathcal{A}}$ even though the statement is similar to Seshadri's ampleness criterion (Theorem 3.3). Note that there is an assumption that a general fiber should be smooth.

(2) Proposition 3.1 is not true when $g = 0$. Indeed, $K_{\overline{\mathcal{M}}_{0,n}} \equiv 2\kappa + \psi$ ([Moo11, Lemma 2.6]). It is well-known that for $n = 4, 5$, $K_{\overline{\mathcal{M}}_{0,n}}$ is anti-ample. So it intersects *negatively* with every curve.

We will use following two positivity results.

Theorem 3.3 (Seshadri's criterion, [Laz04, Theorem 1.4.13]). *Let X be a projective variety and D is a divisor on X . Then D is ample if and only if there exists a positive number $\epsilon > 0$ such that*

$$D \cdot C \geq \epsilon \cdot \text{mult}_x C$$

for every point $x \in C$ and every irreducible curve $C \subset X$.

Theorem 3.4. [Cor93, Lemma 3.2] *There are positive integers h and M depending on r and d , such that the following statement holds for any flat family $\pi : \mathcal{U} \rightarrow B$ of nodal curves over any integral curve B . Let L be a relative degree d line bundle on \mathcal{U} . Suppose that π is not isotrivial as a family of polarized curves. Moreover, assume that*

- (1) *a general fiber is smooth,*
- (2) *$R^1\pi_*(L^i) = 0$ for $i > 0$ and $r := \dim H^0(\mathcal{U}_b, L_{\mathcal{U}_b})$ is independent of $b \in B$,*
- (3) *For a general $b \in B$, $L_{\mathcal{U}_b}$ is base-point-free, very ample and embeds \mathcal{U}_b in \mathbb{P}^{r-1} as a Hilbert stable subscheme.*

Then

$$(3) \quad \begin{aligned} & \left(\frac{r}{2}(L \cdot L) - d(\deg \pi_*(L)) \right) h^2 \\ & + \left((g-1)(\deg \pi_*(L)) - \frac{r}{2}(L \cdot \omega) \right) h + r \deg \lambda \geq \frac{1}{M} \text{mult}_x B \end{aligned}$$

for every point $x \in B$.

Remark 3.5. (1) In [Cor93], Cornalba assumed that $g \geq 2$, but the proof of the theorem shows that this result is true without the assumption. See [Cor93, Section 3] and [CH88, Section 2].

(2) If $d \geq 2g > 0$, then by [HM98, Theorem 4.34] a smooth curve is automatically Hilbert stable. Note that the same proof can be applied to $g = 1$ case, too.

(3) Even if a given family of curves is isotrivial as a family of *abstract* curves, we can apply the theorem if the family is not isotrivial as a family of *polarized* curves.

(4) The equation (3) is different from [Cor93, (3.3)]. But if we follow the proof, the right formula is (3).

Proof of Proposition 3.1. We will divide the proof into several steps.

Step 1. It is sufficient to show the result for a weight datum $n \cdot \tau = (\tau, \tau, \dots, \tau)$ for sufficiently small $\tau > 0$ satisfying $n \cdot \tau \leq 1$.

Let $\varphi_{\mathcal{A}, n \cdot \tau} : \overline{\mathcal{M}}_{g, \mathcal{A}} \rightarrow \overline{\mathcal{M}}_{g, n \cdot \tau}$ be the reduction morphism. By the assumption that a general fiber of π is smooth, $\overline{B} := \varphi_{\mathcal{A}, n \cdot \tau}(B)$ is an integral curve in $\overline{\mathcal{M}}_{g, n \cdot \tau}$. By the projection formula,

$$(4) \quad \begin{aligned} (2\kappa + \psi) \cdot \overline{B} &= \varphi_{\mathcal{A}, n \cdot \tau}^*(2\kappa + \psi) \cdot B = (2\kappa + 2 \sum_{w_I > 1} D_{0, I} + \psi - |I| \sum_{w_I > 1} D_{0, I}) \cdot B \\ &= (2\kappa + \psi) \cdot B - (|I| - 2) \sum_{w_I > 1} D_{0, I} \cdot B \leq (2\kappa + \psi) \cdot B, \end{aligned}$$

because $|I| \geq 2$. Thus if the result is true for the weight datum $n \cdot \tau$, then

$$(2\kappa + \psi) \cdot B \geq (2\kappa + \psi) \cdot \overline{B} \geq e_{g, n \cdot \tau} \cdot \text{mult}_{\varphi_{\mathcal{A}, n \cdot \tau}(x)} \overline{B} \geq e_{g, n \cdot \tau} \cdot \text{mult}_x B.$$

Therefore if we define $e_{g, \mathcal{A}} := e_{g, n \cdot \tau}$, the proposition holds.

Step 2. We can reduce the number of sections.

Let $\rho : \overline{\mathcal{M}}_{g, n \cdot \tau} \rightarrow \overline{\mathcal{M}}_{g, (n-1) \cdot \tau}$ be the forgetful morphism. There are two possible cases. If $\overline{B} := \rho(B)$ is a curve, then

$$(2\kappa + \psi) \cdot \overline{B} = \rho^*(2\kappa + \psi) \cdot B = (2\kappa + \psi) \cdot B.$$

Thus $(2\kappa + \psi) \cdot B = (2\kappa + \psi) \cdot \overline{B} \geq e_{g, (n-1) \cdot \tau} \cdot \text{mult}_{\pi(x)} \overline{B} \geq e_{g, (n-1) \cdot \tau} \cdot \text{mult}_x B$.

If $\rho(B)$ is a point, then the family $\pi : \mathcal{U} \rightarrow B$ is isotrivial as a family of abstract curves after forgetting the last section. For $g \geq 2$, we will use Theorem 3.4 with $L = \omega^k(\sigma_n)$ for sufficiently large k . Then $d = 2k(g-1) + 1$, $r = (2k-1)(g-1) + 1$ and L satisfies all assumptions in Theorem 3.4. Note that $(\pi : \mathcal{U} \rightarrow B, L)$ is not isotrivial as a family of polarized curves.

By the Riemann-Roch theorem,

$$\deg \pi_*(L) = \frac{(L \cdot L)}{2} - \frac{(L \cdot \omega)}{2} + \deg \lambda,$$

since $R^1\pi_*(L) = 0$. Note that in our situation, $\deg \lambda = \lambda \cdot B = 0$ and $D_{\text{nod}} \cdot B = 0$, $\psi_i \cdot B = 0$ for $i = 1, 2, \dots, n-1$. Thus $\kappa \cdot B = 0$ by Mumford's relation and $\psi \cdot B = \psi_n \cdot B$. So it is straightforward to check that

$$\begin{aligned} & \left(\frac{r}{2}(L \cdot L) - d(\deg \pi_*(L)) \right) h^2 + \left((g-1)(\deg \pi_*(L)) - \frac{r}{2}(L \cdot \omega) \right) h + r \deg \lambda \\ &= \frac{gh(h-1) + 2h}{2} \psi_n. \end{aligned}$$

Therefore

$$(2\kappa + \psi) \cdot B = \psi_n \cdot B \geq \alpha \cdot \text{mult}_x B$$

for some $\alpha > 0$ by Theorem 3.4.

When $g = 1$, let $L = \mathcal{O}(k\sigma_n)$ for sufficiently large k . Then $d = r = k$ and L satisfies all assumptions in Theorem 3.4. In this case,

$$\begin{aligned} & \left(\frac{r}{2}(L \cdot L) - d(\deg \pi_*(L)) \right) h^2 + \left((g-1)(\deg \pi_*(L)) - \frac{r}{2}(L \cdot \omega) \right) h + r \deg \lambda \\ &= \frac{k^2 h(h-1)}{2} \psi_n. \end{aligned}$$

So we can find $\epsilon_{g,n,\tau} > 0$ by taking the minimum of these cases.

Step 3. For $\overline{\mathcal{M}}_{g,(\tau)} \cong \overline{\mathcal{M}}_{g,1}$, the proposition holds.

First of all, suppose that $g \geq 2$. Let $\rho : \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ be the forgetting morphism. If $\overline{B} = \rho(B)$ is a curve, then

$$(2\kappa + \psi) \cdot B = 2\rho^*(\kappa) \cdot B + \psi \cdot B = 2\kappa \cdot \overline{B} + \pi_*(\omega \cdot \sigma_1).$$

The divisor κ is ample on $\overline{\mathcal{M}}_g$ by [CH88, Theorem 1.3]. By Seshadri's criterion (Theorem 3.3), there is a positive number $\alpha > 0$ such that $\kappa \cdot \overline{B} \geq \alpha \cdot \text{mult}_x \overline{B}$ for every irreducible curve \overline{B} and $x \in \overline{B}$.

On the other hand, let $\pi' : \mathcal{U}' \rightarrow \overline{B}$ be the corresponding family of stable curves. Then there is a stabilization morphism $\tilde{\rho} : \mathcal{U} \rightarrow \mathcal{U}'$ and $\omega = \tilde{\rho}^*(\omega) + E$ where E is an exceptional curve. Now

$$\pi_*(\omega \cdot \sigma_1) = \pi^*((\tilde{\rho}^*(\omega) + E) \cdot \sigma_1) > 0$$

because ω is ample on \mathcal{U}' by [Ara71, Proposition 3.2], and $E \cdot \sigma_1 > 0$.

If $\pi : \mathcal{U} \rightarrow B$ is isotrivial after forgetting the section σ_1 , then by exactly same argument in Step 2, we can get the inequality (2).

On $\overline{\mathcal{M}}_{1,1}$, $\kappa = 0$ and $\psi_1 = \frac{1}{12}\lambda$ is ample ([AC98, Theorem 2.2], note that κ_1 in [AC98] is $\kappa + \psi_1$.) Therefore we obtain $\epsilon_{1,(1)} > 0$ and the inequality (2) by Seshadri's criterion. \square

4. PROOF OF THE MAIN THEOREM

In this section, we prove our main result.

Theorem 4.1. *Let $\mathcal{A} = (a_1, a_2, \dots, a_n)$ be a weight datum satisfying $2g - 2 + \sum_{i=1}^n a_i > 0$. Then*

$$\overline{\mathcal{M}}_{g,n}(K_{\overline{\mathcal{M}}_{g,n}} + 11\lambda + \sum_{i=1}^n a_i \psi_i) \cong \overline{\mathcal{M}}_{g,\mathcal{A}}$$

where $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is the coarse moduli space of the moduli space $\overline{\mathcal{M}}_{g,\mathcal{A}}$ of \mathcal{A} -stable curves.

Remark 4.2. (1) Theorem 4.1 is a generalization of [Moo11, Theorem 1.4] because when $g = 0$, the Hodge class λ is trivial.

(2) Theorem 4.1 suggests that there is an unexpected relation between log canonical model of moduli spaces and that of parameterized curves. Giving a theoretical reason of this phenomenon would be interesting.

A key step of the proof is to construct an ample divisor on $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

Proposition 4.3. *Let*

$$\Delta_{\mathcal{A}} := K_{\overline{\mathcal{M}}_{g,\mathcal{A}}} + 11\lambda + \sum_{i=1}^n a_i \psi_i = 2\kappa + \sum_{i=1}^n (1 + a_i) \psi_i.$$

Then the push-forward $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is ample.

Proof. By using definitions of tautological divisors and several formulas in Section 2, it is straightforward to see that

$$\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) = 2\kappa + \sum_{i=1}^n (1 + a_i) \psi_i + \sum_{w_{\{i,j\}} \leq 1} w_{\{i,j\}} D_{i=j} = \pi_*((\omega + \sum_{i=1}^n a_i \sigma_i)(2\omega + \sum_{i=1}^n \sigma_i)).$$

A key feature of $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ is that if we restrict it to boundaries, the result is also described the same formula. More precisely, by Lemma 2.6, it is straightforward to check

$$(5) \quad \eta_{i,I}^*(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \pi_1^*(\varphi_{\mathcal{A}_I*}(\Delta_{\mathcal{A}_I})) + \pi_2^*(\varphi_{\mathcal{A}_{I^c}*}(\Delta_{\mathcal{A}_{I^c}})).$$

Also by Lemma 2.7,

$$(6) \quad \xi^*(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \varphi_{\mathcal{A} \cup \{1,1\}*}(\Delta_{\mathcal{A} \cup \{1,1\}}).$$

Finally, for $I \subset [n]$ such that $w_I \leq 1$,

$$(7) \quad \chi_I^*(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \varphi_{\mathcal{A}'*}(\Delta_{\mathcal{A}'}) + (|I| - 1)\pi_*((\omega + \sum_{i \in J} a_i \sigma_i) \cdot \sigma_p)$$

where J is the index set for new weight datum \mathcal{A}' (See the notation for Lemma 2.8.).

We will use Seshadri's criterion (Theorem 3.3) to show the ampleness of $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$. For $\overline{\mathcal{M}}_{1,1}$, it is straightforward to check the ampleness, and $g = 0$ is shown in [Moo11]. So we can use the induction on the dimension of $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

If B is contained in a boundary of nodal curves, then $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) \cdot B \geq \epsilon \cdot \text{mult}_x B$ by (5) and (6). If B is in a boundary of coincident sections, then by the induction hypothesis,

$\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}'})$ is ample and $\pi_*((\omega + \sum_{i \in J} a_i \sigma_i) \cdot \sigma_p)$ is nef by the third equation of [Fed11, Theorem 3.1]. Thus $\chi_1^*(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}))$ is ample by (7) and we can find $\epsilon > 0$.

So it is sufficient to check for the case that $B \cap \mathcal{M}_{g,\mathcal{A}} \neq \emptyset$. Let $\pi : \mathcal{U} \rightarrow B$ with $\sigma_i : B \rightarrow \mathcal{U}$ for $i = 1, 2, \dots, n$ be the family of \mathcal{A} -stable curves. We rewrite $\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})$ as

$$\begin{aligned} \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) &= \varphi_{\mathcal{A}*}((\omega + \sum_{a_i=1} \sigma_i + \sum_{a_i < 1} a_i \sigma_i)(2\omega + \sum_{i=1}^n \sigma_i)) \\ &= \varphi_{\mathcal{A}*}(((1-\delta)\omega + \sum_{a_i=1} (1-\delta)\sigma_i + \sum_{a_i < 1} a_i \sigma_i)(2\omega + \sum_{i=1}^n \sigma_i)) \\ &\quad + \delta(\sum_{a_i=1} \sigma_i)(2\omega + \sum_{i=1}^n \sigma_i) + \delta\omega(2\omega + \sum_{i=1}^n \sigma_i) \\ &= \varphi_{\mathcal{A}*}(((1-\delta)\omega + \sum_{a_i=1} (1-\delta)\sigma_i + \sum_{a_i < 1} a_i \sigma_i)(2\omega + \sum_{i=1}^n \sigma_i)) + \delta \sum_{a_i=1} \psi_i + \delta(2\kappa + \psi). \end{aligned}$$

Note that for there is $\delta > 0$ which depends on g and \mathcal{A} such that

$$\omega + \sum_{a_i=1} \sigma_i + \sum_{a_i < 1} \frac{1}{1-\delta} a_i \sigma_i$$

satisfies the assumption of [Fed11, Proposition 2.1]. So it is nef on \mathcal{U} . By [Moo11, Lemma 3.4], $2\omega + \sum_{i=1}^n \sigma_i$ is effective. Thus

$$\varphi_{\mathcal{A}*}(((1-\delta)\omega + \sum_{a_i=1} (1-\delta)\sigma_i + \sum_{a_i < 1} a_i \sigma_i)(2\omega + \sum_{i=1}^n \sigma_i))$$

is nef on B . For the forgetful map $\rho : \overline{\mathcal{M}}_{g,\mathcal{A}} \rightarrow \overline{\mathcal{M}}_g$, let $\pi' : \mathcal{U}' \rightarrow \rho(B)$ be the corresponding family and σ'_i be the image of section σ_i on \mathcal{U}' . Then $\psi_i = -\sigma_i^2 \geq -\sigma_i'^2 = \omega \cdot \sigma'_i \geq 0$ by [Ara71, Proposition 3.2]. Finally, by Proposition 3.1, there exists $\epsilon > 0$ depends only on g and \mathcal{A} such that

$$\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) \cdot B \geq \epsilon \cdot \text{mult}_x B$$

for all $x \in B$.

There are only *finitely* many boundary strata on $\overline{\mathcal{M}}_{g,\mathcal{A}}$. Therefore we can find the minimum of ϵ for all strata of $\overline{\mathcal{M}}_{g,\mathcal{A}}$ and we obtain an $\epsilon > 0$ for all curves in $\overline{\mathcal{M}}_{g,\mathcal{A}}$. \square

Now Theorem 4.1 is an immediate consequence of Proposition 4.3.

Proof of Theorem 4.1. By Corollary 2.4, it is straightforward to check that

$$\Delta_{\mathcal{A}} = \varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}}) + \sum_{w_I \leq 1} (|I| - 2)(1 - w_I) D_{0,I}.$$

Note that $D_{0,I}$ with $|I| \geq 3$ and $w_I \leq 1$ is an exceptional divisor for $\varphi_{\mathcal{A}}$. Therefore $\Delta_{\mathcal{A}}$ is a sum of the pull-back of an ample divisor and a $\varphi_{\mathcal{A}}$ -exceptional effective divisor. Hence we obtain

$$\overline{\mathcal{M}}_{g,n}(\Delta_{\mathcal{A}}) = \overline{\mathcal{M}}_{g,n}(\varphi_{\mathcal{A}}^* \varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \overline{\mathcal{M}}_{g,\mathcal{A}}(\varphi_{\mathcal{A}*}(\Delta_{\mathcal{A}})) = \overline{\mathcal{M}}_{g,\mathcal{A}}.$$

See [Moo11, Proof of Theorem 3.1] for more detail. \square

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